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Instabilities of convection rolls with stress-free boundaries near threshold

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The stability properties of steady two-dimensional solutions describing convection in a horizontal fluid layer heated from below with stress-free boundaries are investigated in the neighbourhood of the critical Rayleigh number. The region of stable convection rolls as a function of the wavenumber α and the Rayleigh number R is bounded towards higher α by the monotonic skewed varicose instability, while towards low wavenumbers stability is limited by the zigzag instability or by the oscillatory skewed varicose instability. Only for a limited range of Prandtl numbers, $0.543 < P < \infty$, does a finite domain of stability exist. In particular, convection rolls with the critical wavenumber α_c are always unstable.

1. Introduction

Convection in a horizontal layer heated from below is widely regarded as the simplest example of hydrodynamic instability, and thus has been a subject of intense study in the context of the problem of the transition to turbulence. Among the various boundary conditions that can be considered the case of stress-free top and bottom boundaries with infinite heat conductivity is distinguished because it gives rise to simple solutions in the form of trigonometric functions. In particular, the vertical dependence of the motion is independent of the horizontal wavenumber α in the linear approximation of the problem. While the problem of convection with stress-free boundaries has received much attention for its mathematical simplicity, it is also of interest from the experimental point of view. Experimental realizations are possible (Goldstein & Graham 1963) and could become of increasing importance as the novel phenomena associated with the limit of stress-free boundaries become more widely recognized.

Nonlinear studies of convection with stress-free boundaries have been popular among theoretical fluid dynamicists because the results appeared to be qualitatively similar to those for rigid boundaries, which are more difficult to obtain. In both cases convection in the form of rolls represents the only stable steady solution at low Rayleigh number and the instabilities restricting the region of stability in the Rayleigh-number–wavenumber (R,α)-space seemed to be similar. Siggia & Zippelius (1981) pointed out, however, that the zigzag instability involves a nearly undamped component of vertical vorticity in the case of stress-free boundaries, while rigid boundaries require as much viscous friction for components of motions associated with vertical vorticity as for other components. In two subsequent papers (Zippelius & Siggia 1982, 1983) the authors also studied the skewed varicose instability which occurs near the critical point (R_c, α_c) for the onset of convection in the (R,α)-space for stress-free boundaries, while the corresponding stability boundary in the case of rigid boundaries is removed from the critical point. The stability analysis applied by Zippelius & Siggia, however, includes some restrictive assumptions about the wavenumbers of the disturbances, and thus does not capture all mechanisms of instability.

In this paper the stability of convection rolls with respect to disturbances causing small deviations from the two-dimensional structure of motion is analysed for small amplitudes A of convection. Oscillatory onset as well as monotonic onset of instability are considered. A new instability, the oscillatory skewed varicose instability, is isolated and new stability boundaries for the monotonic skewed varicose stability are derived. The results suggest that convection with stress-free boundaries exhibits a number of new phenomena not found in the case of rigid boundaries.

The basic equation for the complex growth rate σ of disturbances is derived in §2. Several different cases of instability are analysed in §3, and the stability regime of convection rolls is discussed in §4. The paper closes with some remarks on general aspects of the problem in §5.

2. Formulation of the mathematical problem

We consider a horizontal convection layer of height h with the temperatures T_1 and T_2 prescribed at the upper and lower boundaries. Using h as lengthscale, h^2/ν as timescale, where ν is the kinematic viscosity of the fluid, and $(T_2-T_1)R^{-1}$ as temperature scale, we can write the Boussinesq equations for the velocity vector $\boldsymbol{v}$ and the heat equation for the deviation θ of the temperature from the static distribution in the dimensionless forms

$$\nabla\cdot\boldsymbol{v} = 0, \tag{2.1a}$$

$$\nabla^2\boldsymbol{v}+\boldsymbol{k}\theta-\nabla\pi = \boldsymbol{v}\cdot\nabla\boldsymbol{v}+\frac{\partial}{\partial t}\boldsymbol{v}, \tag{2.1b}$$

$$\nabla^2\theta+R\boldsymbol{k}\cdot\boldsymbol{v} = P\left(\boldsymbol{v}\cdot\nabla\theta+\frac{\partial}{\partial t}\theta\right), \tag{2.1c}$$

where $\boldsymbol{k}$ is the unit vector opposite to the direction of gravity, and Rayleigh and Prandtl numbers are defined as usual:

$$R = \frac{\gamma g(T_2-T_1)h^3}{\nu\kappa}, \quad P = \frac{\nu}{\kappa}.$$

After introducing the general representation for a solenoidal vector field

$$\boldsymbol{v} = \nabla\times(\nabla\times\boldsymbol{k}\phi)+\nabla\times\boldsymbol{k}\psi \equiv \boldsymbol{\delta}\phi+\boldsymbol{\epsilon}\psi \tag{2.2}$$

we formulate the problem in terms of ϕ and ψ by operating with $\boldsymbol{\delta}\cdot$ and $\boldsymbol{\epsilon}\cdot$ onto equation (2.1 b)

$$\nabla^4\Delta_2\phi-\Delta_2\theta = \boldsymbol{\delta}\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v})+\frac{\partial}{\partial t}\nabla^2\Delta_2\phi, \quad \nabla^2\Delta_2\psi = \boldsymbol{\epsilon}\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v})+\frac{\partial}{\partial t}\Delta_2\psi, \tag{2.3a,b}$$

where $\Delta_2 \equiv \nabla^2-(\boldsymbol{k}\cdot\nabla)^2$ is the horizontal Laplacian. A Cartesian system of coordinates will be used in the following, with the z-coordinate in the direction of $\boldsymbol{k}$ and the origin on the lower boundary of the layer. The conditions of vanishing tangential stress and infinite conductivity at the boundaries can be expressed in the form

$$\phi = \frac{\partial^2}{\partial z^2}\phi = \frac{\partial}{\partial z}\psi = \theta = 0 \quad \text{at } z = 0, 1. \tag{2.4}$$

According to the analysis of Schlüter, Lortz & Busse (1965) the only possibly stable steady solution of (2.1*c*), (2.3) amd (2.4) near the critical value R_c of the Rayleigh number is the two-dimensional solution

$$\phi = A\cos\alpha x \sin\pi z + \dots, \tag{2.5a}$$

$$\theta = (\pi^2+\alpha^2)^2\left(\phi - \frac{\alpha^2 PA^2}{8\pi}\sin 2\pi z + \dots\right), \tag{2.5b}$$

$$R = \frac{(\pi^2+\alpha^2)^3}{\alpha^2} + \tfrac{1}{8}(PA\alpha)^2(\pi^2+\alpha^2)^2 + \dots, \tag{2.5c}$$

$$\psi = 0, \tag{2.5d}$$

where terms of order A^n, $n \geqslant 3$, have not been given explicitly.

To investigate the stability of the steady solution (2.5) we superimpose infinitesimal disturbances of the form

$$\tilde{\phi} = \{\tilde{\phi}_0 + A\tilde{\phi}_1 + A^2\tilde{\phi}_2 + \dots\}\exp\{\mathrm{i}dx + \mathrm{i}by + \sigma t\}, \tag{2.6a}$$

$$\tilde{\psi} = \{\tilde{\psi}_0 + A\tilde{\psi}_1 + A^2\tilde{\psi}_2 + \dots\}\exp\{\mathrm{i}dx + \mathrm{i}by + \sigma t\}, \tag{2.6b}$$

$$\tilde{\theta} = \{\tilde{\theta}_0 + A\tilde{\theta}_1 + A^2\tilde{\theta}_2 + \dots\}\exp\{\mathrm{i}dx + \mathrm{i}by + \sigma t\} \tag{2.6c}$$

where $\tilde{\phi}_n$, $\tilde{\psi}_n$ and $\tilde{\theta}_n$ are functions of x and z with the same period in x as the steady solution (2.5). The equations to be satisfied by the disturbances (2.6) are

$$\nabla^4\Delta_2\tilde{\phi} - \Delta_2\tilde{\theta} = \boldsymbol{\delta}\cdot[\boldsymbol{\delta}\phi\cdot\nabla(\boldsymbol{\delta}\tilde{\phi}+\boldsymbol{\epsilon}\tilde{\psi}) + (\boldsymbol{\delta}\tilde{\phi}+\boldsymbol{\epsilon}\tilde{\phi})\cdot\nabla\boldsymbol{\delta}\phi] + \sigma\nabla^2\Delta_2\tilde{\phi}, \tag{2.7a}$$

$$\nabla^2\Delta_2\tilde{\psi} = \boldsymbol{\epsilon}\cdot[\boldsymbol{\delta}\phi\cdot\nabla(\boldsymbol{\delta}\tilde{\phi}+\boldsymbol{\epsilon}\tilde{\psi}) + (\boldsymbol{\delta}\tilde{\phi}+\boldsymbol{\epsilon}\tilde{\phi})\cdot\nabla\boldsymbol{\delta}\phi] + \sigma\Delta_2\tilde{\psi}, \tag{2.7b}$$

$$\nabla^2\tilde{\theta} - R\Delta_2\tilde{\phi} = P\{\boldsymbol{\delta}\phi\cdot\nabla\tilde{\theta} + (\boldsymbol{\delta}\tilde{\phi}+\boldsymbol{\epsilon}\tilde{\psi})\cdot\nabla\theta + \sigma\tilde{\theta}\}. \tag{2.7c}$$

We are interested in those disturbances that cause only small deviations from the two-dimensional structure of the steady solution (2.5). We shall thus restrict attention to small values of the wavenumbers d and b. In addition there are disturbances with $(d^2+b^2)^{\frac{1}{2}} \approx \alpha$, which will not be considered here since no new results beyond those given by Schlüter *et al.* (1965) and Busse (1971) are expected.

Anticipating that growing disturbances have growth rates σ of order A or smaller, we find the following solutions for $\tilde{\phi}_0$, $\tilde{\theta}_0$ and $\tilde{\psi}_0$ which approximately satisfy (2.7) with vanishing right-hand side:

$$\tilde{\phi}_0 = \sin\pi z \sum_{n=1}^{2} c^{(n)}\exp\{(-1)^n\,\mathrm{i}\alpha x\}, \tag{2.8a}$$

$$\tilde{\theta}_0 = \sin\pi z \sum_{n=1}^{2} c^{(n)}(\pi^2+(\alpha+(-1)^n d)^2+b^2)^2\exp\{(-1)^n\,\mathrm{i}\alpha x\}, \tag{2.8b}$$

$$\tilde{\psi}_0 = 0. \tag{2.8c}$$

A finite value of $\tilde{\psi}_0$ could be assumed since (2.7*b*) and (2.4) admit the possibility of slowly decaying horizontal flows with vanishing z-dependence. But, since we assume that such flows do not occur in the absence of steady convection, the expansion for $\tilde{\psi}$ starts with the term $\tilde{\psi}_1$. For the average of $\tilde{\psi}_1$ over the fluid layer, $c^{(3)} \equiv \langle\tilde{\psi}_1\rangle$, (2.7*b*) yields

$$(b^2+d^2+\sigma)(d^2+b^2)c^{(3)} = -\sum_{n=1}^{2}\tfrac{1}{4}c^{(n)}[2d\alpha+(-1)^n(d^2+b^2)]\alpha b\pi^2. \tag{2.9}$$

The growth rate σ has been included in (2.9) since it may be of same order of magnitude as d^2+b^2. For $\tilde{\phi}_1$ and $\tilde{\theta}_1$ the following expressions are obtained from (2.7*a*, *c*):

$$\tilde{\phi}_1 = \sin 2\pi z \frac{\alpha^2}{64\pi^3} \sum_{n=1}^{2} c^{(n)} \left[4\alpha^2 - 2\beta(\alpha^2-\pi^2) + \frac{P(\pi^2+\alpha^2)^2}{\pi^2} + o(d) \right], \tag{2.10a}$$

$$\tilde{\theta}_1 = \sin 2\pi z \frac{P(\pi^2+\alpha^2)\,\alpha^2}{4\pi} \sum_{n=1}^{2} c^{(n)} \left[\pi^2+\alpha^2 - \frac{(-1)^n d(\pi^2+3\alpha^2)}{\alpha} + o(d^2, b^2) \right], \tag{2.10b}$$

where β is defined by

$$\beta = b^2/(d^2+b^2) \tag{2.10c}$$

and terms of order d in (2.10*a*) and of order d^2 or b^2 in (2.10*b*) have not been given explicitly because they will not be needed in the following arguments.

After inserting (2.9) and (2.10) on the right-hand side of (2.7*a*, *c*) we can obtain equations for the unknown $c^{(n)}$ with σ as eigenvalue. By multiplying (2.7*a*) by $\Delta_{(m)}^{-4} R^{(m)} \exp\{i[(-1)^m \alpha - d]x - iby\}$ and (2.7*c*) by $\Delta_{(m)}^{-2} \exp\{i[(-1)^m \alpha - d)x - iby\}$, $m = 1, 2$, adding the two equations and averaging them over the fluid layer, we get

$$\sum_{n=1}^{2} c^{(n)} \left\{ [\sigma(1+P) - r^{(n)}] \Delta^{-2} \delta_{nm} + G\left[1 - \frac{d}{\alpha}(\eta_1(-1)^m + \eta_2(-1)^n) \right] \right\}$$
$$+ (-1)^m \tfrac{1}{2} b\alpha(1+P)\left(1 - (-1)^m \eta_3 \frac{d}{\alpha}\right) c^{(3)}$$
$$= 0 \quad \text{for } m = 1, 2, \tag{2.11}$$

where the following definitions have been used,

$$\left.\begin{aligned} \Delta_{(n)} &\equiv \pi^2 + (\alpha - (-1)^n d)^2 + b^2, \\ R^{(n)} &\equiv \Delta_{(n)}^3 / [(\alpha - (-1)^n d)^2 + b^2], \\ r^{(n)} &\equiv [(\pi^2+\alpha^2)^3/\alpha^2 - R^{(n)}]\, \Delta_{(n)}^{-2} [(\alpha - (-1)^n d)^2 + b^2] \end{aligned}\right\} \quad \text{for } n = 1, 2, \tag{2.12}$$

$$G \equiv \tfrac{1}{8} P^2 \alpha^4, \quad \eta_1 = -\tfrac{5}{6} - \frac{1}{8}\left(1 - \frac{5}{3P}\right)\left(P^{-1} + \frac{\beta}{2P} + \tfrac{9}{8}\right),$$

$$\eta_2 = \tfrac{5}{3}, \quad \eta_3 = \frac{1}{3}\left(-4 + \frac{2}{1+P}\right).$$

Terms of higher order in d and b than those retained in (2.11) have been neglected since they are not needed in the following analysis. In evaluating η_1, η_2 and η_3 the property has been used that the wavenumber α of the steady solution is close to its critical value $\alpha_c = \pi/\sqrt{2}$. The difference between α and its critical value is only of interest in the expression $r^{(n)}$, which for small values of $\alpha - \alpha_c$, d and b is given by

$$r^{(n)} = -4\left[-(-1)^n (\alpha - \alpha_c)\, d \Big/ \left(2 - \frac{\alpha - \alpha_c}{\alpha}\right) + d^2\left(1 - \frac{\alpha - \alpha_c}{3\alpha}\right) + (\alpha - \alpha_c)\frac{b^2}{\alpha} + \frac{b^4}{4\alpha^2} \right]. \tag{2.13}$$

Equations (2.9) and (2.11) represent a system of three homogeneous equations for the unknowns $c^{(n)}$, $n = 1, 2, 3$, which can be solved if and only if the determinant of the coefficient matrix vanishes. This latter condition gives rise to a cubic equation for σ,

$$
\begin{aligned}
(\sigma+b^2+d^2)\Bigg\{&\sigma^2(1+P)^2-\sigma(1+P)\,(r^{(1)}+r^{(2)}-2A^2G)+r^{(1)}r^{(2)} \\
&-A^2G\left[r^{(1)}+r^{(2)}-\frac{(\eta_1+\eta_2)\,(r^{(1)}-r^{(2)})\,d}{\alpha}\right]-\frac{A^4G^2\eta_1\eta_2 d^2}{\alpha^2}\Bigg\} \\
&+b^2(1+P)\frac{\alpha^2\pi^2}{8}A^2\sum_{n=1}^{2}\left(1+\frac{(-1)^n\eta_3 d}{\alpha}\right)\Bigg\{\left(1+\frac{(-1)^n 2\alpha d}{b^2+d^2}\right) \\
&\times\left[\sigma(1+P)+A^2G\left(1-\frac{(-1)^n(\eta_1+\eta_2)\,d}{\alpha}\right)-r^{(n)}\right] \\
&+\left(1-\frac{(-1)^n 2\alpha d}{d^2+b^2}\right)A^2G\left(1-\frac{(-1)^n(\eta_1-\eta_2)\,d}{\alpha}\right)\Bigg\}=0,
\end{aligned}
\tag{2.14}
$$

which will be discussed in the following section.

3. Growth-rate analysis

In analysing the roots of (2.14) it is illuminating to start with the limit $d=b=0$ for which the simple equation

$$\sigma\{\sigma^2(1+P)^2+\sigma(1+P)\,2A^2G\}=0 \tag{3.1}$$

is obtained. The three roots of this equation,

$$\sigma^{(1)}=-2A^2G(1+P)^{-1},\quad \sigma^{(2)}=\sigma^{(3)}=0, \tag{3.2}$$

correspond to disturbances of the form

$$
\left.
\begin{aligned}
&\tilde{\phi}^{(1)}=\phi,\quad \tilde{\theta}^{(1)}=\theta,\quad \tilde{\psi}^{(1)}=0,\\
&\tilde{\phi}^{(2)}=\frac{\partial\phi}{\partial x},\quad \tilde{\theta}^{(2)}=\frac{\partial\theta}{\partial x},\quad \tilde{\psi}^{(2)}=0,\\
&\tilde{\phi}^{(3)}=0,\quad \tilde{\psi}^{(3)}\neq 0.
\end{aligned}
\right\}
\tag{3.3}
$$

The changes of the eigenvalues $\sigma^{(2)}$ and $\sigma^{(3)}$ introduced by small values of d and b are the subject of this section.

We first consider the limit of vanishing d, in which case (2.14) reduces to

$$
\begin{aligned}
\sigma^3(1+P)^2+\sigma^2(1+P)\,[b^2(1+P)+4A^2G\xi]+\sigma[\xi^2+2\xi(A^2G+(1+P)\,b^2) \\
+b^2(1+P)\,(2G+F)\,A^2+b^2[\xi^2+\xi(2G+F)\,A^2+2GFA^4]=0,
\end{aligned}
\tag{3.4a}
$$

where the definitions

$$\xi\equiv 4(\alpha-\alpha_c)\frac{b^2}{\alpha}+\frac{b^4}{\alpha^2},\quad F=\tfrac{1}{4}(1+P)\,\alpha^2\pi^2$$

have been used. For $\xi>0$ all coefficients of σ^n in (3.4) are positive and all real roots σ are negative. The first coefficient that changes sign as ξ decreases is the last term on the left-hand side of (3.4). The real root σ which becomes positive can be best studied if the assumption $|\sigma|\ll b^2$ is made. From the general relationship (2.14) the equation

$$\sigma^2(1+P)^2+2\sigma(1+P)\,(\xi+GA^2+\tfrac{1}{2}FA^2)+(\xi+2GA^2)\,(\xi+FA^2)=0 \tag{3.4b}$$

follows in this case, which yields the roots

$$\sigma_1 = \frac{-\xi - 2GA^2}{1+P}, \quad \sigma_2 = \frac{-\xi - FA^2}{1+P}. \tag{3.5a, b}$$

The maximum value of σ as a function of b is achieved for $b^2 = 2(\alpha_c - \alpha)\alpha$ in both cases (3.5). The conditions for instabilities corresponding to positive growth rates (3.5) are thus given by

$$R - R_c < (R^{(0)} - R_c)\left[1 + \frac{P^2}{4(1+P)}\right], \quad R - R_c < \tfrac{3}{2}(R^{(0)} - R_c) \text{ for } \alpha < \alpha_c, \tag{3.6a, b}$$

where R_c is the minimum value of $R^{(0)} \equiv (\pi^2 + \alpha^2)^3/\alpha^2$. In the limit $P \to \infty$ (3.6a) is identical with the zigzag instability criterion derived by Schlüter *et al.* (1965). But the conclusions of the latter paper are not correct in the case of finite Prandtl number and stress-free boundaries. Siggia & Zippelius (1981) first pointed out the important role played by the ψ-component of the velocity field at finite Prandtl numbers and derived criterion (3.6a) for finite Prandtl number. Criterion (3.6b) corresponds to a new mechanism of instability. In contrast to the zigzag instability which corresponds to modified disturbances of the second type in (3.3), the new wavy mode represents a modification of disturbances of the first type in (3.3). But because the Eckhaus instability, to be discussed below, always precedes it, the new wavy instability is unlikely to be important.

So far the assumption has been made that all roots of (3.4) are real. It can be shown that for sufficiently small values of b two conjugate complex roots exist. These roots have been investigated by Busse (1972). The application of the latter analysis to the present problem demonstrates that the real parts of the complex roots are negative as will also become apparent in the analysis described below. Only when $R - R_c$ exceeds a finite value of the order P^{-2} does the oscillatory instability occur.

In the limit of vanishing b the summation term on the left-hand side of (2.14) vanishes, and in lowest order the familiar Eckhaus criterion for instability with respect to two-dimensional disturbances is recovered:

$$R - R_c < 3(R^{(0)} - R_c). \tag{3.7a}$$

By taking into account terms of the next higher order, the asymmetry of the stability boundary with respect to $\alpha - \alpha_c$ can be determined. A refined version of the condition for instability with respect to two-dimensional distrubances is thus obtained:

$$R - R_c < 18\pi^2(\alpha - \alpha_c)^2\left[3 - \frac{\alpha - \alpha_c}{\alpha}\left(\frac{97}{144} - \frac{7}{16P} - \frac{5}{6P^2}\right)\right]. \tag{3.7b}$$

In the general case of (2.14), $b \neq 0 \neq d$, it is convenient to obtain expressions for σ in terms of the expansion

$$\sigma = \sigma_{10} b + \sigma_{01} d + \sigma_{20} b^2 + \sigma_{11} db + \sigma_{02} d^2 + \ldots, \tag{3.8}$$

where the constant terms vanishes because the attention is restricted to modifications of the growth rates $\sigma^{(2)}$, $\sigma^{(3)}$. The detailed inspection of (2.14) shows that σ_{01} and σ_{11} also vanish, since all terms linear in d cancel. For the coefficient σ_{10} the following relationship is obtained:

$$\sigma_{10}^2 + \tfrac{1}{4}a^2\pi^2 A^2\left[1 - \left(\frac{10}{3} + \frac{8(\alpha - \alpha_c)\alpha}{A^2 G}\right)(1 - \beta)\right] = 0, \tag{3.9}$$

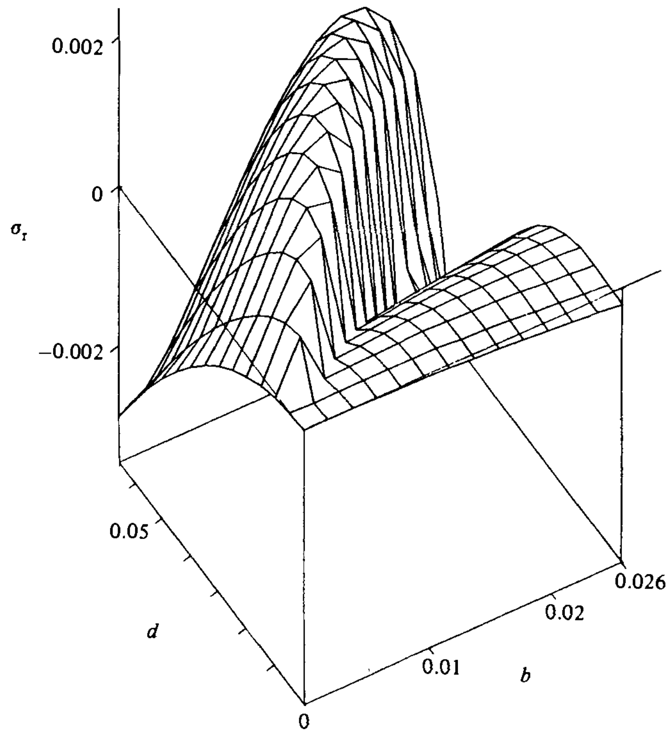

FIGURE 1. The real part σ_r of the growth rate as a function of d and b for $R = 670$, $\alpha = 2.188$ and $P = 0.71$.

which indicates instability if for sufficiently small values of β the condition

$$(\alpha-\alpha_c)\,\alpha+\tfrac{7}{6}A^2G > 0 \tag{3.10a}$$

is satisfied. The criterion (3.10a) for the onset of the monotonic skewed varicose instability can be expressed in terms of the Rayleigh number R

$$R-R_c > \alpha(\alpha_c-\alpha)\tfrac{108}{7}\pi^2, \tag{3.10b}$$

the form of which indicates that the wavenumber α of rolls must decrease sufficiently strongly with increasing Rayleigh number in order to avoid this instability. A remarkable property is the vanishing dependence on the Prandtl number. Except for this latter property, it resembles the skewed varicose instability in the case of convection with rigid boundaries (Busse & Clever 1979), which also occurs for small values of b and d in the neighbourhood of the stability boundary. As the stability boundary is approached, the maximum growth rate is achieved for $b \approx d^{\frac{3}{2}}$, which follows from (3.8) if the result $\sigma_{02} < 0$ is used. This property is evident from figure 1, where σ has been plotted as a function of d and b for an unstable point in the (R, α) parameter space.

In addition to the real roots of (3.9), there exist complex roots if $(\alpha_c-\alpha)/A^2$ becomes sufficiently large or if β approaches unity. The latter case is of lesser interest since the limit $\beta = 1$ yields the oscillatory mode which has been mentioned above and which does not lead to instability in the neighbourhood of the critical Rayleigh number for finite Prandtl numbers. In the former case, however, a new instability occurs, which we call the oscillatory skewed varicose instability. In order to determine the region of growth of this instability, terms of higher order in the expansion (3.8) must be evaluated. Since σ_{11} vanishes we obtain for

$$\sigma_2 \equiv \sigma_{20} + \sigma_{02} d^2/b^2 \tag{3.11}$$

the following relationship from (2.14),

$$\frac{\sigma_2 P^2}{1+P} = (1-\beta) M - N\beta^{-1} + 2P^2(1+P)^{-2}, \tag{3.12}$$

where the definitions

$$M = \frac{8(\alpha_c - \alpha)\alpha}{A^2 G} - \frac{2(3+P)}{3(1+P)}, \quad N = \tfrac{1}{2}P^2(1+P)^{-2}(5+P) \tag{3.13a, b}$$

have been introduced. It is easily seen that positive values σ_2 can only be obtained for $M > 0$, implying $\alpha < \alpha_c$. The maximum of σ_2 is reached for

$$\beta^2 = N/M, \tag{3.14}$$

yielding the condition
$$M > \frac{P^2[3+P+(5+P)^{\frac{1}{2}}(1+P)^{\frac{1}{2}}]}{(1+P)^2} \tag{3.15}$$

for instability. This inequality together with relationship (3.14) implies

$$(d/b)^2 > (1+P)^{\frac{1}{2}}(5+P)^{-\frac{1}{2}}, \tag{3.16}$$

which shows that the condition $\beta \leqslant 1$ required by (2.10c) is satisfied. We note that σ_2 is negative for $\beta^2 = 1$, i.e. $d = 0$, as we have anticipated above. Criterion (3.15) for instability with respect to the oscillatory skewed instability can be cast into a more convenient form by expressing A^2 in terms of $R - R^{(0)}$:

$$R - R^{(0)} < \frac{(\alpha_c - \alpha)\alpha\, 36\pi^2(1+P^{-1})^2}{(3+P)(\tfrac{2}{3}P^{-2}(1+P)+1) + (5+P)^{\frac{1}{2}}(1+P)^{\frac{1}{2}}} \tag{3.17}$$

for instability.

4. Stability region of convection rolls

In contrast with the case of rigid boundaries, where three stability boundaries meet at the point (R_c, α_c) of the (R, α)-space, there are five stability boundaries meeting at this point in the case of stress-free boundaries. It is thus not surprising that the range of stable rolls is much more severely curtailed in the present case. Indeed there exists a critical Prandtl number P_c below which no stable steady convection flow is possible in the neighbourhood of the point (R_c, α_c). This phenomenon occurs when R and α satisfy either criterion (3.10b) or criterion (3.17) for instability, and P_c is determined by

$$1 = \left(\frac{P}{1+P}\right)^2 \left[5 - \frac{4}{P} + 3P + 3(P^2+6P+5)^{\frac{1}{2}}\right], \tag{4.1}$$

which yields the value
$$P_c = 0.543. \tag{4.2}$$

Zippelius & Siggia (1982, 1983) find a new instability for $P < 0.782$, which resembles the monotonic skewed varicose instability, but corresponds to relatively large values of d such that the disturbance wavevector lies in the region where the conducting state is stable. The place where the stability boundary corresponding to this new modified skewed varicose instability meets the Eckhaus stability boundary determines

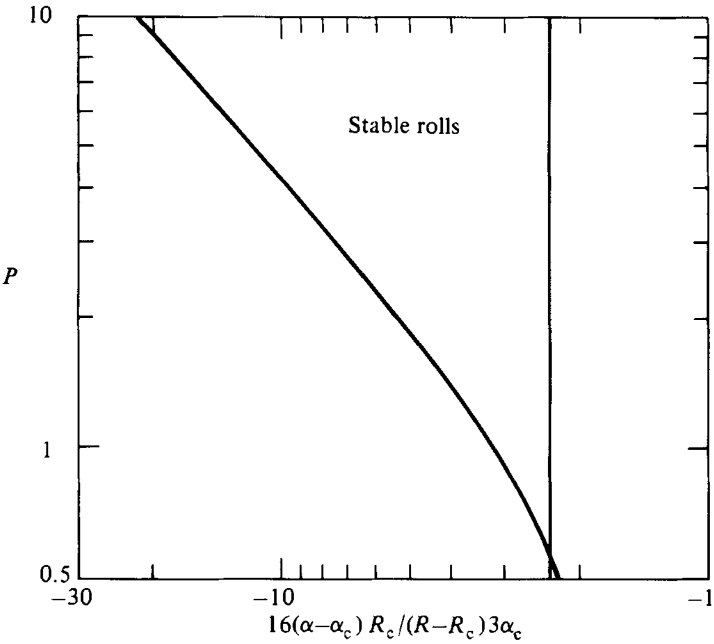

FIGURE 2. The tangents of the angles that define the boundaries of the wedge-like area within which rolls are stable in the (R, α)-plane are shown as a function of the Prandtl number P. The angle of the stability boundary for monotonic skewed varicose instability is independent of P; the curve on the left side of the figure corresponds to the oscillatory skewed varicose instability.

in their analysis the Prandtl number below which no stable convection rolls exist. Neither in the present analysis nor in the numerical computations of Bolton & Busse (1984) was evidence for the modified skewed varicose of Zippelius & Siggia found – apparently because it disappears as terms of higher order than those included by Zippelius & Siggia are taken into account. Terms of higher order not included by Zippelius & Siggia also change the properties of the ordinary monotonic skewed varicose instability. Zippelius & Siggia derive the criterion $\alpha > \alpha_c$ for the onset of this instability, while the more accurate criterion is given by inequality (3.10*b*).

The region of stable rolls also tends to disappear in the limit of large Prandtl number. The intersection of the zigzag stability boundary with the skewed varicose stability boundary is responsible for this property. According to (3.6) and (3.10*b*), rolls are unstable at any value of α for

$$\frac{R-R_c}{R_c} \geqslant \frac{192(1+P)}{49P^2}. \tag{4.3}$$

Of course this inequality is strictly valid only for small values of $R-R^{(0)}$ and thus applies for large Prandtl numbers $P \approx R_c/(R-R_c)$.

For finite Prandtl numbers the stability region of convection rolls is a scissor-like wedge opening to the left of α_c in the (R, α)-space as P exceeds P_c. The Prandtl-number dependence of the angle of this wedge is shown in figure 2. Finite-amplitude computations, which will be reported elsewhere (Bolton & Busse 1984), show that the two stability boundaries forming the wedge tend to become parallel as R increases. For large Prandtl numbers the parabolic boundary of the zigzag instability closes in rapidly and limits the region of stable Rayleigh number according to criterion (4.3).

5. Concluding remarks

In the analysis described in this paper a minimal use of formal asymptotic expansions has been made, since the small parameters of the problem, such as A, b, d and $\alpha-\alpha_c$, enter in the equations in a variety of different ways. A more rigorous mathematical formulation of the problem would have led to a very cumbersome analysis. The main omission concerns terms of higher order in A, primarily those of order A^4, because terms of odd powers in A do not enter expressions for the growth rates. Terms of order A^4 are not important, however, since they do not change the property that the growth rates $\sigma^{(1)}$, $\sigma^{(2)}$ vanish in the limit $d, b \rightarrow 0$. All results derived in this paper have been checked against independently derived numerical results, which do not depend on restrictions on the amplitude of convection. The numerical stability analysis of the convection rolls has been carried out in close analogy to the corresponding analysis for rigid boundaries (Clever & Busse 1974; Busse & Clever 1979). Besides providing an independent confirmation of the analytical results reported in this paper, the numerical analysis to be reported in a separate paper (Bolton & Busse 1984) extends these results to high Rayleigh numbers.

The most surprising result of the present analysis is the phenomenon that convection rolls setting in at the critical Rayleigh number become unstable as soon as the Rayleigh number increases beyond the critical value. The skewed varicose instability may lead to a change of the wavelength of the rolls, and steady rolls may be realizable for a limited range of Rayleigh numbers provided that the Prandtl number exceeds the critical value P_c. Experimental evidence about the skewed varicose instability in the case of rigid boundaries suggests a different scenario. As Gollub, McCarriar & Steinman (1982) have demonstrated, random variations of the convection pattern on a slow timescale are introduced when the boundary of instability with respect to skewed varicose disturbances is reached. Convection with stress-free boundary conditions offers the attractive possibility of investigating this phenomenon in the weakly nonlinear limit of the basic equations.

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